Consider a test of

$H_0: p = p_0$ vs $H_A: p > p_0$

$p < p_0$

$p \neq p_0$

where we wish to have a Type I error rate of α.

Then $P(\text{rej } H_0 \mid H_0 \text{ true}) = \alpha$

$= P(\hat{p} \in \mathcal{C} \mid p = p_0)$

where $\mathcal{C}$ is chosen to give level α.

Recall that if $X_i \overset{iid}{\sim} B(1,p)$ are n trials where we let $X=1$ when we observe a "success", then

$Y = \sum_{i=1}^{n} X_i$ is the total number of "successes"

$Y \sim B(n,p)$ by mgf

So $\hat{p} = \bar{X}$ is the proportion of "successes" in n trials

We note that

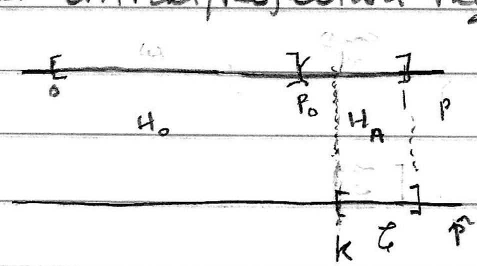
$E(\bar{X}) = p \qquad Var(\bar{X}) = \frac{p(1-p)}{n}$

By the CLT, for n "large" $(np(1-p) \geq 5)$

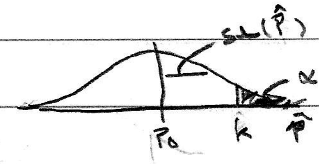
$\hat{p} = \bar{X} \,\dot{\sim}\, N(p, \frac{p(1-p)}{n})$

To test $H_0: p \leq p_0$ vs $H_A: p > p_0$

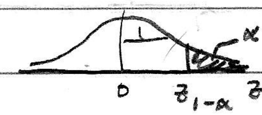
our critical/rejection region will be large $\hat{p}$



We need



$$\begin{aligned}\alpha &= P(\hat{P} > k \mid p = p_0)\\ &= P\left(\frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > \frac{k - p_0}{\sqrt{p_0(1-p_0)/n}}\right)\\ &= P(Z > z_{1-\alpha})\\ &= 1 - P(Z \le z_{1-\alpha})\end{aligned}$$



We reject H_0 for

$$\frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > z_{1-\alpha}$$

where $Z \sim N(0,1)$.

These days we just compute the p-value.

$$\text{p-value} = \begin{cases} P(\hat{P} \ge \hat{p} \mid p = p_0) & H_A: p > p_0 \\ 2P(|\hat{P} - p_0| \ge |\hat{p} - p_0| \mid p = p_0) & H_A: p \ne p_0 \\ P(\hat{P} \le \hat{p} \mid p = p_0) & H_A: p < p_0 \end{cases}$$

We reject when p-value $< \alpha$.

Ex: (NFL coin flip)

At the start of overtime in NFL games they flip a coin to see which team kicks and which receives. Does the flip winner have an advantage?

Between 1974 and 2009 the flip winner won 240 of 428 games.

Assume the 1974 to 2009 overtime games are a sample from all possible overtime games. (BIG assumption since coaches learn)

Do the data provide sufficient evidence to conclude that the flip winner has an advantage?

Sol: $H_0: p \le .5$ vs $H_A: p > .5$ p = prop of game wins for flip winner

Sample prop is $\hat{p} = 240/428 = .56$ $n = 428$

Is n "large"? $n p_0 = 428(.5) = 214 > 10$ yes

Testsat $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} = \dfrac{.56 - .5}{\sqrt{\dfrac{.5(1-.5)}{n}}}$

$= 2.483$

StatKey → normal → $N(0,1)$ → right tail → 2.483

p-value = .0065

Rej H_0 and conclude that there is evidence that the flip winners win a higher proportion of games.

Ex (Mendel's peas)

According to Father Gregor Mendel's (1822-1884) theory of genetics, under a certain set of conditions the proportion of plants producing smooth green peas should be $p = \frac{3}{16} = .1875$. A sample of 556 plants from his experiment had 108 with smooth green peas. Do the data refute Mendel's ideas?

Sol: Test $H_0: p = .1875$ vs $H_A: p \neq .1875$

p = prop of plants w/ smooth green peas

$\hat{p} = 108/556 = .1942$ $\quad n = 556$

Is n "large"? $np_0 = 556(.1875) = 104.25 > 10$ yes

Test stat is

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{.1942 - .1875}{\sqrt{\frac{.1875(1-.1875)}{556}}}$$

$$= .405$$

StatKey → normal → N(0,1) → two-tail → .405

p-value $= 2(.343) = .686$

Do not rej H_0. There is insufficient evidence to refute Mendel's claim.

This does not mean Mendel was right (we might make a Type II error)

	SG	Sg	sG	sg
SG	SG	SG	SG	SG
Sg	SG	Sg	SG	Sg
sG	SG	SG	sG	sG
sg	SG	Sg	sG	sg

$\frac{1}{4}$ smooth recessive

$\frac{3}{4}$ Green Dominant

$$\frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$$

Testing the pop mean

Consider a test of

$H_0: \mu = \mu_0$ vs $H_A: \mu > \mu_0$

$\mu < \mu_0$

$\mu \neq \mu_0$

where we wish to have a Type I error rate of α.

Then $P(\text{rej } H_0 \mid H_0 \text{ true}) = \alpha$

$= P(\bar{x} \in C \mid \mu = \mu_0)$

where C is chosen to gives level α.

Recall that for n "large" the CLT says

$\bar{X} \overset{\cdot}{\sim} N(\mu, \frac{\sigma^2}{n})$

If σ^2 is known then

$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \overset{\cdot}{\sim} N(0,1)$

If σ^2 is unknown and n is "large" $(n \geq 30)$

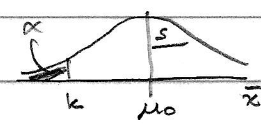
or $X_i \overset{iid}{\sim} N(\mu_0, \sigma^2)$ then it can be shown that

$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \overset{\cdot}{\sim} t_{n-1}$

To test $H_0: \mu \geq \mu_0$ vs $H_A: \mu < \mu_0$

We need $\alpha = P(\bar{X} < k \mid \mu = \mu_0)$

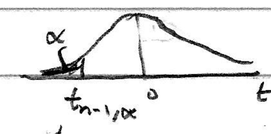
$= P\left(\frac{\bar{X} - \mu_0}{s/\sqrt{n}} < \frac{k - \mu_0}{s/\sqrt{n}}\right)$



we reject for $t < t_{n-1,\alpha}$

$\frac{\bar{x} - \mu_0}{s/\sqrt{n}} < t_{n-1,\alpha}$

$\bar{x} < \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}}$



note $t_{n-1,\alpha} = -t_{n-1,1-\alpha} < 0$

Ex (Dark Chocolate)

Eleven people were given 4.6g (1.6 oz) of dark chocolate per day for two weeks, and their vascular health was measured before and after the two weeks. The mean health increase for the subjects was 1.3 with standard deviation 2.3. Larger values indicate better health. Do the data support the claim that dark chocolate improves vascular health? Use $\alpha = .05$.

Sol: X = vascular health improvement from eating choc for two weeks

$H_0: \mu \le 0 = \mu_0$ vs $H_A: \mu > 0 = \mu_0$ improves

$n = 11$ so $df = n-1 = 10$

$\bar{x} = 1.3$ $s = 2.3$

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{1.3 - 0}{2.3/\sqrt{11}} = \frac{1.3}{.6935} = 1.8746$$

$P(T > 1.8746) = .045$

Since $.045 < .05$ we rej H_0 and conclude that there is evidence that dark chocolate improved vascular health.